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Propagation of Waves Through Cylindrical Bore in a Cubic Micropolar Generalized Thermoelastic Medium

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Abstract The article deals with the propagation of axial symmetric cylindrical surface waves in a cylindrical bore through a micropolar thermoelastic medium of infinite extent possessing cubic symmetry. The theories of generalized thermoelasticity developed by Lord and Shulman and Green and Lindsay are used to study the problem. The frequency equations, connecting the phase velocity with the wave number, radius of bore, and other material parameters for empty and liquid-filled bores are derived. Some special cases have been deduced. The numerical results obtained have been illustrated graphically to understand the behavior of the phase velocity and attenuation coefficient versus the wave number.

Keywords Attenuation coefficient · Cylindrical bore · Generalized thermoelasticity · Micropolar · Phase velocity

1 Introduction

In the classical theory of elasticity the microstructure of a material is not taken into consideration for studying the mechanical behavior of the material due to external stimuli. But discrepancies are observed in the classical theory and experimental results while studying the stress concentration in the neighborhood of the holes and cracks, especially in a material containing laminates, granules, and fibers. The discrepancies indicate that the material response to external stimuli depends on the motions of inner structures, and so a study of a micropolar elastic medium is necessary.

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Eringen and Suhubi [1] and Suhubi and Eringen [2] developed a non-linear theory of microelastic solids and microfluids in which the micromotions of the material particles contained in a macro-volume element with respect to its centroid are taken into account in an average sense. Materials affected by such micromotions and microdeformations are called micromorphic materials.

Eringen [3,4] developed theories for a subclass of micromorphic materials which are called micropolar media, and these materials show microrotational effects and microrotational inertia. Here, the material particles in a volume element can undergo only rigid rotational motions about their center of mass. The motion described here is not only by a deformation but also by a microrotation giving six degrees of freedom. The interaction between two parts of a body is transmitted not only by a force but also by a torque, resulting in asymmetric force stresses and coupled stresses. Physically, solid propellant grains, polymeric materials, and fiber glass are examples of such materials. The theory is expected to find applications in the treatment of mechanics of granular materials, composites fibrous materials, and particularly microcracks and microfractures.

The dynamical interactions between thermal and mechanical fields in solids has great practical applications in aeronautics, nuclear reactors, and high energy particle accelerators. The classical theory of heat conduction predicts an infinite speed of heat transport, if a material conducting heat is subjected to a thermal disturbance, which contradicts the physical facts. During the last three decades non-classical theories have been developed to remove this paradox.

The generalized theory of thermoelasticity was developed by Lord and Shulman [5] involving one relaxation time for isotropic homogeneous media, which is called the first generalization to the coupled theory of elasticity. These equations determine the finite speeds of propagation of heat and displacement distributions; the corresponding equations for the anisotropic case were obtained by Dhaliwal and Sherief [6].

The second generalization to the coupled theory of elasticity is what is known as the theory of thermoelasticity, with two relaxation times or the theory of temperature-dependent thermoelasticity. A generalization of this inequality was proposed by Green and Laws [7]. Green and Lindsay [8] obtained an explicit version of the constitutive equations. These equations were also obtained independently by Suhubi [9]. This theory contains two constants that act as relaxation times and modify not only the heat equations, but also all the equations of the coupled theory. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry.

The linear theory of micropolar thermoelasticity was developed by Nowacki [10] and Eringen [11] by extending the theory of micropolar continua to include thermal effects. Tauchert et al. [12] also derived the basic equations of the linear theory of micropolar thermoelasticity. Dost and Tabarrok [13] presented micropolar generalized thermoelasticity by using the Green–Lindsay theory. Chandrasekhariah [14] formulated a theory of micropolar thermoelasticity which includes a heat flux, among the constitutive variables.

The propagation of waves in thermoelastic micropolar materials has many applications in various fields of science and technology, namely, atomic physics, industrial engineering, thermal power plants, submarine structures, pressure vessels, aerospace, chemical pipes, and metallurgy. The importance of thermal stresses in causing structural damages and changes in functioning of the structure is well recognized whenever thermal stress environments are involved.

A cubic anisotropic medium possess three independent elastic constants compared to two for isotropic media which are often assumed in classical elasticity. One result of the additional constant is a coupling of terms in the Navier equations for a cubic medium. A wide class of crystals such as W, Si, Cu, Ni, Fe, Au, and Al, which are frequently used substances, belong to the cubic materials. The cubic materials have nine planes of symmetry whose normals are on the three co-ordinate axes and on the co-ordinate planes making an angle of $\pi/4$ with the co-ordinate axes. With the chosen co-ordinate system along the crystalline directions, the mechanical behavior of a micropolar cubic crystal can be characterized by four independent elastic constants.

The problem of propagation of waves along a cylindrical bore embedded in an infinite micropolar thermoelastic medium possessing cubic symmetry is of great importance due to its manifold applications. In practice, the cylindrical bore may be realized by a borehole or a mine gallery. Borehole studies are of great interest in exploration seismology, e.g., in the exploration of oils, gases, hydrocarbons, etc. In the oil industry, acoustic borehole logging is commonly practiced. A borehole is drilled in a potential hydrocarbon reservoir and then probed with an acoustic tool. Almost all oil companies rely on seismic interpretation for selecting the sites for exploratory oil wells. Seismic wave methods also have higher accuracy, higher resolutions, and are more economical as compared to drilling which is costly and time consuming.

Tomar and Kumar [15], Deswal et al. [16], and Kumar et al. [17] studied problems of wave propagation through a cylindrical bore in a micropolar elastic medium with a stretch and micropolar elastic medium.

In the present article we have discussed the propagation of surface waves near a cylindrical bore hole through a micropolar generalized thermoelastic medium possessing cubic symmetry. Frequency equations relating the phase velocity and wave number are derived for an empty as well as for a liquid-filled bore. The dispersion curves giving the phase velocity and attenuation coefficient as functions of the wave number are plotted for different values of the radius for an anisotropic as well as for an isotropic case.

2 Basic Equations

Following Passarella [18] and Green and Lindsay [8], the constitutive relations and balance laws in a generalized thermomicropolar anisotropic medium possessing a center of symmetry, in the absence of body forces and body couples are given by
Constitutive relations:

$$t_{ij} = C_{ijkl}E_{kl} + G_{ijkl}\Psi_{kl} + A_{ij}\left(1 + t_1\frac{\partial}{\partial t}\right)T,$$

$$m_{ij} = G_{klji}E_{kl} + \Gamma_{jikl}\Psi_{kl} + G_{ij}\left(1 + t_1\frac{\partial}{\partial t}\right)T,$$

$$\rho\eta = -A_{ij}E_{ij} - G_{ij}\Psi_{ij} + \frac{\rho C_e}{T_0}T,$$
$$t_0\dot{q}_i = K_{ij}T_{,j} - q_{i.}$$

The deformation and wryness tensor are defined by

$$E_{ji} = u_{i,j} + \varepsilon_{ijk}\phi_k, \qquad \Psi_{ij} = \phi_{i,j}. \tag{1}$$

Balance laws:

$$\begin{aligned} t_{ij,j} &= \rho\ddot{u}_i, \\ m_{ij,j} - \varepsilon_{irs}t_{rs} &= \rho j\ddot{\phi}_i, \\ q_{i,i} &= \rho T_0\dot{\eta}, \end{aligned} \tag{2}$$

where t_{ij}, m_{ij}, and K_{ij} are, respectively, the stress tensor, couple stress tensor, and thermal conductivity tensor, q_i is the heat flux vector, η is the entropy, T is the absolute temperature, C_e is the specific heat at constant strain, t_0 and t_1 are thermal relaxation times, ρ is the bulk mass density, j is the microinertia, and u_i, ϕ_i are, respectively, the components of the displacement vector and microrotation vector. $C_{ijkl}, G_{ijkl}, \Gamma_{ijkl}, A_{ij}$, and G_{ij} are characteristic constants of the material following the symmetry properties given by Passarella [18].

3 Problem Formulation and Its Solution

We have used appropriate transformations, following Atanackovic et al. [19], on the set of Eqs. 1 to derive equations for a generalized micropolar thermoelastic medium possessing cubic symmetry. We consider a cylindrical bore of radius a having a circular cross-section in a generalized micropolar thermoelastic medium possessing cubic symmetry. We use cylindrical polar coordinates (r, θ, z) with the z-axis pointing upward along the axis of the cylinder. The propagation of axial symmetric waves is considered near the borehole, and these waves are the analog of Rayleigh waves propagating at a traction free boundary of a generalized micropolar thermoelastic medium possessing cubic symmetry. This section deals with the situation when the bore does not contain any fluid. We are discussing a two-dimensional problem with symmetry about the z-axis, so all partial derivatives with respect to the variable θ would be zero. Therefore, we take $\vec{u} = (u_r, 0, u_z)$, $\vec{\phi} = (0, \phi_\theta, 0)$, and $\partial/\partial\theta \equiv 0$, so that all the field equations and constitutive relations in cylindrical polar coordinates reduce to

$$\frac{\partial t_{rr}}{\partial r} + \frac{\partial t_{zr}}{\partial z} + \frac{t_{rr} - t_{\theta\theta}}{r} = \rho\frac{\partial^2 u_r}{\partial t^2}, \tag{3}$$
$$\frac{\partial t_{rz}}{\partial r} + \frac{\partial t_{zz}}{\partial z} + \frac{t_{rz}}{r} = \rho\frac{\partial^2 u_z}{\partial t^2}, \tag{4}$$
$$\frac{\partial m_{r\theta}}{\partial r} + \frac{\partial m_{z\theta}}{\partial z} + \frac{m_{r\theta} + m_{\theta r}}{r} + t_{zr} - t_{rz} = \rho j\frac{\partial^2 \phi_\theta}{\partial t^2} \quad , \tag{5}$$

$$K_1\left(\frac{\partial^2 T}{\partial t^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)=\rho C_e\left(1+t_0\frac{\partial}{\partial t}\right)\dot{T}$$
$$+\nu T_0\left(1+t_0 n_0\frac{\partial}{\partial t}\right)\left(\frac{\partial \dot{u}_r}{\partial r}+\frac{\dot{u}_r}{r}+\frac{\partial \dot{u}_z}{\partial z}\right), \quad (6)$$

$$t_{rr}=C_{11}\frac{\partial u_r}{\partial r}+C_{12}\left(\frac{\partial u_z}{\partial z}+\frac{u_r}{r}\right)-\nu\left(1+t_1\frac{\partial}{\partial t}\right)T, \quad (7)$$

$$t_{rz}=C_{44}\left(\frac{\partial u_z}{\partial r}+\phi_\theta\right)+C_{45}\left(\frac{\partial u_r}{\partial z}-\phi_\theta\right), \quad (8)$$

$$t_{zr}=C_{44}\left(\frac{\partial u_z}{\partial r}-\phi_\theta\right)+C_{45}\left(\frac{\partial u_r}{\partial z}+\phi_\theta\right), \quad (9)$$

$$t_{\theta\theta}=C_{12}\left(\frac{\partial u_r}{\partial r}+\frac{\partial u_z}{\partial z}\right)+C_{11}\frac{u_r}{r}-\nu\left(1+t_1\frac{\partial}{\partial t}\right)T, \quad (10)$$

$$t_{zz}=C_{12}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)+C_{11}\frac{\partial u_z}{\partial z}-\nu\left(1+t_1\frac{\partial}{\partial t}\right)T, \quad (11)$$

$$m_{r\theta}=\Gamma_{44}\frac{\partial\phi_\theta}{\partial r}-\Gamma_{45}\frac{\phi_\theta}{r}, \quad (12)$$

$$m_{\theta r}=\Gamma_{45}\frac{\partial\phi_\theta}{\partial r}-\Gamma_{44}\frac{\phi_\theta}{r}, \quad (13)$$

$$m_{z\theta}=\Gamma_{44}\frac{\partial\phi_\theta}{\partial z}. \quad (14)$$

where $A_{ij}=-\nu\delta_{ij}$, $\nu=(C_{11}+2C_{12})\alpha_T$, and α_T is the coefficient of linear thermal expansion; we have used the notations $11\to 1, 22\to 2, 12\to 4, 21\to 5$ for the material constants.

For the Lord and Shulman (L–S) theory, $t_1=0, n_0=1$, and for the Green and Lindsay (G–L) theory, $t_1>0, n_0=0$. The thermal relaxation times t_0 and t_1 satisfy the inequality $t_1\geq t_0>0$ for the G–L theory only. However, it has been proved by Sturnin [20] that the inequalities are not mandatory for t_0 and t_1 to follow.

For further considerations, it is convenient to introduce dimensionless variables defined by

$$\{r',z'\}=\frac{1}{a}\{r,z\},\quad \{u'_r,u'_z\}=\frac{\rho\omega_1 c_1}{\nu T_0}\{u_r,u_z\},\quad \phi'_\theta=\frac{\rho c_1^2}{\nu T_0}\phi_\theta,\quad t'_{ij}=\frac{t_{ij}}{\nu T_0},$$
$$m'_{r\theta}=\frac{\omega_1}{c_1\nu T_0}m_{r\theta},\quad T'=\frac{T}{T_0},\quad c_1^2=\frac{C_{11}}{\rho},\quad \{t',t'_0,t'_1\}=\frac{c_1}{a}\{t,t_0,t_1\},$$
$$\omega_1=\frac{\rho C_e c_1^2}{K_1} \quad (15)$$

Using dimensionless variables defined by Eq. 15, in Eqs. 3–6 with the help of Eqs. 7 –14, after suppressing the primes, the field equations in cylindrical coordinates reduce to

$$\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)+a_1\frac{\partial^2 u_r}{\partial z^2}+a_2\frac{\partial^2 u_z}{\partial r\partial z}-a_3\frac{\partial \phi_\theta}{\partial z}-a_4\left(1+t_1\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial r}$$
$$=\frac{\partial^2 u_r}{\partial t^2}, \tag{16}$$

$$a_1\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+\frac{\partial^2 u_z}{\partial z^2}+a_2\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)+a_3\left(\frac{\partial \phi_\theta}{\partial r}+\frac{\phi_\theta}{r}\right)$$
$$-a_4\left(1+t_1\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial z}=\frac{\partial^2 u_z}{\partial t^2}, \tag{17}$$

$$\frac{\partial^2 \phi_\theta}{\partial r^2}+\frac{\partial^2 \phi_\theta}{\partial z^2}+\frac{1}{r}\frac{\partial \phi_\theta}{\partial r}-\frac{\phi_\theta}{r^2}+a_5\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)-2a_6\phi_\theta=a_7\frac{\partial^2 \phi_\theta}{\partial t^2}, \tag{18}$$

$$\nabla^2 T=a_8(\dot{T}+t_0\ddot{T})+\varepsilon\left(\frac{\dot{u}_r}{r}+\frac{\partial \dot{u}_r}{\partial r}+\frac{\partial \dot{u}_z}{\partial z}+t_0 n_0\left(\frac{\ddot{u}_r}{r}+\frac{\partial \ddot{u}_r}{\partial r}+\frac{\partial \ddot{u}_z}{\partial z}\right)\right), \tag{19}$$

$$a_1=\frac{C_{44}}{C_{11}},\quad a_2=\frac{C_{12}+C_{45}}{C_{11}},\quad a_3=\frac{a\omega_1(C_{44}-C_{45})}{c_1 C_{11}},\quad a_4=\frac{a\rho\omega_1 c_1}{C_{11}},$$
$$a_5=\frac{ac_1(C_{44}-C_{45})}{\omega_1\Gamma_{44}},\quad a_6=\frac{a^2(C_{44}-C_{45})}{\omega_1\Gamma_{44}},\quad a_7=\frac{\rho j c_1^2}{\Gamma_{44}},\quad a_8=\frac{\rho C_e a c_1}{K_1},$$
$$\varepsilon=\frac{\nu^2 T_0}{\rho\omega_1 K_1}. \tag{20}$$

Assuming the solutions of Eqs. 16–19 for the waves propagating in the z-direction as

$$\{u_r,u_z,\phi_\theta,T\}=\{b_1K_1(mr),b_2K_0(mr),b_3K_1(mr),b_4K_0(mr)\}\mathrm{e}^{\mathrm{i}(kz-\omega t)} \tag{21}$$

where $K_0(), K_1()$ are, respectively, the modified Bessel functions of order zero and one and of the second kind, $\omega(=kc)$ is the angular velocity of the wave, k is the wave number, and c is the phase velocity.

Substituting Eq. 21 into Eqs. 16–19, we obtain a system of four homogeneous linear equations in four unknowns b_1, b_2, b_3, and b_4;

$$\begin{aligned}&(m^2-a_1k^2+\omega^2)b_1-\mathrm{i}ka_2mb_2-\mathrm{i}ka_3b_3+a_4(1-\mathrm{i}\omega t_1)mb_4=0,\\&-\mathrm{i}ka_2mb_1+(a_1m^2-k^2+\omega^2)b_2-a_3mb_3-a_4(1-\mathrm{i}\omega t_1)\mathrm{i}kb_4=0,\\&\mathrm{i}ka_5b_1+a_5mb_2+(-m^2-k^2-2a_6+a_7\omega^2)b_3+0.b_4=0,\\&-\varepsilon\mathrm{i}\omega m(1-i\omega t_0n_0)b_1-\varepsilon\omega k(1-\mathrm{i}\omega t_0n_0)b_2+0.b_3\\&\quad+(m^2-k^2+\mathrm{i}\omega a_8+a_8\omega^2)b_4=0.\end{aligned} \tag{22}$$

For the non-trivial solution of Eq. 22, the determinant of coefficients of b_1, b_2, b_3, and b_4 must vanish, which gives

$$m^8+Am^6+Bm^4+Cm^2+D=0, \tag{23}$$

where

$$
\begin{aligned}
A &= -\frac{1}{a_1}(-a_1(\omega^2 - a_1k^2) + a_1a_9 - (\omega^2 - k^2) + a_3a_5 - k^2a_2^2.\\
&\quad - a_1a_4\mathrm{i}\omega\varepsilon(1 - \mathrm{i}\omega t_1)(1 - \mathrm{i}\omega t_0n_0)),\\
B &= -\frac{1}{a_1}\{(\omega^2 - a_1k^2)(a_1a_9 - \omega^2 + k^2 + a_3a_5) + a_1a_{10} + (\omega^2 - k^2)a_9\\
&\quad + k^2a_2^2a_9 - 2k^2a_2a_3a_5 + a_{11} + k^2a_1a_3a_5 + (1 - \mathrm{i}\omega t_0n_0)(1 - \mathrm{i}\omega t_1)\\
&\quad \times(-\mathrm{i}k^2a_2a_4\omega\varepsilon + \mathrm{i}\omega\varepsilon a_3a_4a_5 - \mathrm{i}\omega k^2a_2a_4\varepsilon - a_1a_4\mathrm{i}\omega\varepsilon(k^2 + 2a_6 - a_7\omega^2)\\
&\quad - a_4\mathrm{i}\omega\varepsilon(\omega^2 - k^2))\},\\
C &= -\frac{1}{a_1}\{(\omega^2 - a_1k^2)(a_1a_{10} + (\omega^2 - k^2)a_9 + a_{11}) + (1 - \mathrm{i}\omega t_0n_0)(1 - \mathrm{i}\omega t_1)\\
&\quad \times(-2\mathrm{i}k^2a_3a_4a_5\varepsilon\omega - (k^2 + 2a_6 - a_7\omega^4)(\mathrm{i}\omega\varepsilon k^2a_2a_4 + \mathrm{i}\omega\varepsilon a_4(\omega^2 - k^2)))\\
&\quad + (\omega^2 - k^2)a_{10} - a_{12} + k^2a_2^2a_{10} - k^2a_3a_5(a_8\omega^2 - k^2 + \mathrm{i}\omega a_8)(a_1 + a_2)\},\\
D &= -\frac{1}{a_1}\{(\omega^2 - a_1k^2)((\omega^2 - k^2)a_{10} - a_{12}) + \mathrm{i}\omega\varepsilon(1 - \mathrm{i}\omega t_0n_0)(1 - \mathrm{i}\omega t_1)\\
&\quad \times(k^2a_2a_4(a_7\omega^2 - k^2 - 2a_6) + k^4a_3a_4a_5) - k^2a_3a_5(a_2 + \omega^2 - k^2)\\
&\quad \times(a_8\omega^2 + \mathrm{i}\omega a_8 - k^2)\},\\
a_9 &= a_7\omega^2 - 2a_6 - \mathrm{i}\omega a_8 - a_8\omega^2,\\
a_{10} &= k^4 + 2a_6k^2 - (a_7 + a_8)\omega^2k^2 - \mathrm{i}\omega a_8k^2 - 2\mathrm{i}\omega a_6a_8 + \mathrm{i}\omega^3a_7a_8\\
&\quad - 2a_6a_8\omega^2 + a_7a_8\omega^4,\\
a_{11} &= a_3a_5(a_8\omega^2 + i\omega a_8 - k^2) + \mathrm{i}\omega\varepsilon k^2a_4(1 - \mathrm{i}\omega t_0n_0)(1 - \mathrm{i}\omega t_1),\\
a_{12} &= \mathrm{i}\omega\varepsilon k^2a_4(1 - \mathrm{i}\omega t_0n_0)(1 - \mathrm{i}\omega t_1)(a_7\omega^2 - k^2 - 2a_6).
\end{aligned}
$$

The roots of Eq. 23 are complex in general. These roots are denoted by $m_i^2, i = 1, \ldots, 4$. Corresponding to these roots, the waves with amplitudes b_1, b_2, b_3, and b_4 which are designated by $b_1(i)$, $b_2(i)$, $b_3(i)$, and $b_4(i)$. These are given by

$$
b_1(i) = \frac{\Delta_1(i)}{\Delta(i)}, \quad b_2(i) = \frac{\Delta_2(i)}{\Delta(i)}, \quad b_3(i) = \frac{\Delta_3(i)}{\Delta(i)}, \quad b_4(i) = \frac{\Delta_4(i)}{\Delta(i)},
$$

where

$$
\begin{aligned}
\Delta_1(i) &= -a_1m_i^6 + \left(a_1a_9 - \omega^2 + k^2 + a_3a_5\right)m_i^4 + \left(a_1a_{10} + \left(\omega^2 - k^2\right)a_9 + a_{11}\right)\\
&\quad \times m_i^2 - a_{12} + a_{10}(\omega^2 - k^2),\\
\Delta_2(i) &= -ika_2m_i\left(-m_i^4 + a_9m_i^2 + a_{10}\right) + \mathrm{i}ka_3a_5m_i(m_i^2 - k^2 + \mathrm{i}\omega a_8 + a_8\omega^2)\\
&\quad + \varepsilon\omega km_ia_4(1 - \mathrm{i}\omega t_0n_0)(1 - \mathrm{i}\omega t_1)\left(-m_i^2 - k^2 - 2a_6 + a_7\omega^2\right),\\
\Delta_3(i) &= ka_4a_5\varepsilon\omega(1 - \mathrm{i}\omega t_0n_0)(1 - \mathrm{i}\omega t_1)\left(m_i^2 - k^2\right) + \left(m_i^2 - k^2 + \mathrm{i}\omega a_8 + a_8\omega^2\right)\\
&\quad \times\left(-\mathrm{i}ka_5(a_1 + a_2)m_i^2 - k^2 + \omega^2\right),
\end{aligned}
$$

$$\Delta_4(i) = \mathrm{i}\omega\varepsilon(1 - \mathrm{i}\omega t_0 n_0)m_i(-a_3a_5(m_i^2 - k^2) + (m_i^2 + k^2 + 2a_6 - a_7\omega^2) \times (a_1m_i^2 + k^2a_2 - k^2 + \omega^2)).$$

$$\Delta(i) = \sqrt{(\Delta_1(i))^2 + (\Delta_2(i))^2 + (\Delta_3(i))^2 + (\Delta_4(i))^2}. \tag{24}$$

Thus, the appropriate solutions of Eqs. 16–19, corresponding to the wave propagating along the z-axis, are

$$\{u_r, u_z, \phi_\theta, T\} = \sum_{j=1}^{4} \left\{b_1(j)K_1(m_jr) + b_2(j)K_0(m_ir) + b_3(j)K_1(m_jr) + b_4(j)K_0(m_jr)\right\} f(j)\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{25}$$

where $f(j)$ are relative excitation factors.

3.1 Derivation of Frequency Equation

At the interface $r = 1$, the appropriate boundary conditions are

$$t_{rr} = 0, \quad t_{rz} = 0, \quad m_{r\theta} = 0, \quad \frac{\partial T}{\partial r} = 0, \tag{26}$$

where t_{rr} and t_{rz} are the radial and tangential stress components, respectively, and $m_{r\theta}$ is the torsional coupled stress. Making use of Eq. 25 in Eqs. 7, 8, and 12 and then using boundary conditions, Eqs. 26, we obtain four homogeneous equations in four unknowns $f(j)$, $j = 1, \ldots, 4$, elimination of which gives the frequency equation,

$$H_1\Delta' - H_2\Delta'' + H_3\Delta''' - H_4\Delta'''' = 0, \tag{27}$$

where

$$\begin{aligned}
\Delta' &= P_2(S_3M_4 - S_4M_3) - P_3(S_2M_4 - S_4M_2) + P_4(S_2M_3 - S_3M_2),\\
\Delta'' &= P_1(S_3M_4 - S_4M_3) - P_3(S_1M_4 - S_4M_1) + P_4(S_1M_3 - S_3M_1),\\
\Delta''' &= P_1(S_2M_4 - S_4M_2) - P_2(S_1M_4 - S_4M_1) + P_4(S_1M_2 - S_2M_1),\\
\Delta'''' &= P_1(S_2M_3 - S_3M_2) - P_2(S_1M_3 - S_3M_1) + P_3(S_1M_2 - S_2M_1),
\end{aligned}$$

where

$$\begin{aligned}
H_i &= -\{s_1b_1(i)m_i + s_2k^2b_2(i) + (1 - \mathrm{i}\omega t_0)b_4(i)\}K_0(m_i) + (s_2 - s_1)b_1(i)K_1(m_i),\\
P_i &= \{-s_3b_2(i)m_i + \mathrm{i}ks_4b_1(i) + s_5b_3(i)\}\,K_1(m_i),\\
S_i &= -s_6b_3(i)K_0(m_i) - (s_6 + s_7)b_3(i)K_1(m_i),\; M_i = -m_ib_4(i)K_1(m_i),
\end{aligned}$$

$$s_1=\frac{C_{11}}{a\rho\omega_1c_1},\quad s_2=\frac{C_{12}}{a\rho\omega_1c_1},\quad s_3=\frac{C_{44}}{a\rho\omega_1c_1},\quad s_4=\frac{C_{45}}{a\rho\omega_1c_1},$$
$$s_5=\frac{C_{44}-C_{45}}{\rho c_1^2},\quad s_6=\frac{\Gamma_{44}C_e}{ac_1K_1},\quad s_7=\frac{\Gamma_{45}C_e}{ac_1K_1}.$$

Equation 27 determines the dimensionless phase velocity c of axial symmetric surface waves as a function of the dimensionless wave number k and other thermomicropolar parameters of the medium.

4 Propagation of Waves in a Cylindrical Bore Filled with Liquid

Here, we consider the same problem as in the previous section with the additional constraint that the borehole is filled with a homogeneous inviscid liquid. The field equation and constitutive relations for a homogeneous inviscid liquid are

$$\lambda^{\mathrm{L}}\nabla\left(\nabla\cdot\vec{u}^{\mathrm{L}}\right)=\rho^{\mathrm{L}}\frac{\partial^2\vec{u}^{\mathrm{L}}}{\partial t^2},\tag{28}$$
$$t_{ij}^{\mathrm{L}}=\lambda^{\mathrm{L}}\left(\nabla\cdot\vec{u}^{\mathrm{L}}\right)\delta_{ij},\tag{29}$$

where $\vec{u}^{\mathrm{L}}$ is the displacement vector and λ^{L} and ρ^{L} are, respectively, the bulk modulus and density of the liquid. Other symbols have their usual meaning as defined earlier. For a two-dimensional problem, we take

$$\vec{u}^{\mathrm{L}}=\left(u_r^{\mathrm{L}},0,u_z^{\mathrm{L}}\right),\quad \text{and}\quad \frac{\partial}{\partial\theta}=0.\tag{30}$$

The dimensionless variables defined in this case, in addition to those defined by Eqs. 15, are

$$u_r^{\mathrm{L}'}=\left(\frac{\rho\omega_1c_1}{\nu T_0}\right)u_r^{\mathrm{L}},\quad u_z^{\mathrm{L}'}=\left(\frac{\rho\omega_1c_1}{\nu T_0}\right)u_z^{\mathrm{L}},\quad t_{rr}^{\mathrm{L}'}=\frac{t_{rr}^{\mathrm{L}}}{\nu T_0}.\tag{31}$$

We relate the dimensionless displacement components and potential function ϕ^{L} as

$$u_r^{\mathrm{L}}=\frac{\partial\phi^{\mathrm{L}}}{\partial r},\quad u_z^{\mathrm{L}}=\frac{\partial\phi^{\mathrm{L}}}{\partial z}.\tag{32}$$

Making use of Eq. 32 in Eqs. 28 and 29, with the help of Eqs. 30 and 31, after suppressing the primes, yields

$$\nabla^2\phi^{\mathrm{L}}=\delta_1^2\frac{\partial^2\phi^{\mathrm{L}}}{\partial t^2},\tag{33}$$

and

$$t_{rr}^{\mathrm{L}} = \frac{\lambda^{\mathrm{L}}}{a\rho\omega_1 c_1}\nabla^2\phi^{\mathrm{L}}, \tag{34}$$

where

$$\delta_1 = \frac{c_1}{c^{\mathrm{L}}}, c^{\mathrm{L}} = \sqrt{\frac{\lambda^{\mathrm{L}}}{\rho^{\mathrm{L}}}}. \tag{35}$$

The solution of Eq. 14 corresponding to surface waves may be written as

$$\phi^{\mathrm{L}} = A_0 I_0(m_0 r)\mathrm{e}^{\mathrm{i}(kz-\omega t)}. \tag{36}$$

After some simplification, the pressure and radial displacement of the liquid are given by

$$p^{\mathrm{L}} = -t_{rr}^{\mathrm{L}} = s_8\omega^2 A_0 I_0(m_0 r)\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{37}$$

$$u_r^{\mathrm{L}} = m_0 A_0 I_1(m_0 r)\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{38}$$

where

$$s_8 = \frac{\lambda^{\mathrm{L}}\delta_1^2}{a\rho\omega_1 c_1}, m_0^2 = k^2\left(1-\delta_1^2 c^2\right), \tag{39}$$

and $I_0()$, $I_1()$ are modified Bessel functions of the first kind and of order zero and one, respectively.

4.1 Derivation of Frequency Equation

The appropriate boundary conditions for the present situation are

$$t_{rr} = -p^{\mathrm{L}}, \quad t_{rz} = 0, \quad m_{r\theta} = 0, \quad \frac{\partial T}{\partial r} = 0, \quad u_r = u_r^{\mathrm{L}}, \quad \text{at } r = 1. \tag{40}$$

Making use of Eqs. 36–38 and 25 in the boundary conditions of Eqs. 40, and with the help of Eqs. 7, 8, and 12, we obtain five homogeneous equations in five unknowns A_0, $f(1)$, $f(2)$, $f(3)$, and $f(4)$. The condition for the non-trivial solution yields the frequency equation as

$$s_8\omega^2 I_0(m_0)\left\{-b_1(1)K_1(m_1)\Delta' + b_1(2)K_1(m_2)\Delta'' - b_1(3)K_1(m_3)\Delta''' + b_1(4)K_1(m_4)\Delta''''\right\} - m_0 I_1(m_0)\left\{H_1\Delta' - H_2\Delta'' + H_3\Delta''' - H_4\Delta''''\right\} = 0. \tag{41}$$

5 Special Cases

Case 1 The corresponding expressions of frequency equations are obtained,

(a) for the L–S theory, by taking $t_1 = 0$ and $n_0 = 1$,
(b) for the G–L theory, by taking $t_1 \geq t_0 > 0, n_0 = 0$,

in Eqs. 27 and 41 along with the changed values of $m_{i,} (i = 0, 1, \ldots, 4)$.

Subcase: Neglecting the thermal effect, our results agree with those obtained by Banerji and Sengupta [21,22] by changing dimensionless quantities into physical quantities.

Case 2 Taking $C_{11} = \lambda + 2\mu + \kappa, C_{12} = \lambda, C_{44} = \mu + \kappa, C_{45} = \mu, \Gamma_{44} = \gamma, \Gamma_{45} = \beta$, in Eqs. 27 and 41, we obtain the corresponding expressions for an isotropic micropolar generalized thermoelastic medium with the changed values of m_i, $(i = 0, 1, \ldots, 4)$.

6 Numerical Results and Discussion

For numerical computation, we take the following values of relevant parameters for a micropolar thermoelastic medium possessing cubic symmetry as

$$C_{11} = 18 \times 10^{10}\,\mathrm{N \cdot m^{-2}}, C_{12} = 11.7 \times 10^{10}\,\mathrm{N \cdot m^{-2}}, C_{44} = 5.6 \times 10^{10}\,\mathrm{N \cdot m^{-2}},$$
$$C_{45} = 4.3 \times 10^{10}\,\mathrm{N \cdot m^{-2}}, \Gamma_{44} = 1.88 \times 10^{-9}\,\mathrm{N}, \Gamma_{45} = 0.7 \times 10^{-9}\,\mathrm{N}.$$

For comparison with an isotropic micropolar thermoelastic solid, following Eringen [23] and Dhaliwal and Singh [24], we take the following values of relevant parameters for the case of a magnesium crystal-like material as

$$\rho = 1.74 \times 10^{3}\,\mathrm{kg \cdot m^{-3}}, \lambda = 9.4 \times 10^{10}\,\mathrm{N \cdot m^{-2}}, \mu = 4.0 \times 10^{10}\,\mathrm{N \cdot m^{-2}},$$
$$\kappa = 1.0 \times 10^{10}\,\mathrm{N \cdot m^{-2}}, \gamma = 0.779 \times 10^{-9}\,\mathrm{N}, j = 0.2 \times 10^{-19}\,\mathrm{m^2},$$
$$\mathrm{C_e} = 1.04 \times 10^{3}\,\mathrm{J \cdot kg^{-1} \cdot {}^{\circ}C^{-1}}$$

The dimensionless relaxation times are taken as

(i) for L–S theory: $t_0 = 0.2, \quad t_1 = 0.0$,
(ii) for G–L theory: $t_0 = 0.2, \quad t_1 = 0.6$.

For the dimensionless coupling constant, ε, we have taken the hypothetical value 0.073. Equations 27 and 41 determine the phase velocity, c, of the axial symmetric surface waves as a function of the wave number k, the radius of the bore, and various physical parameters in complex form. If we write

$$\frac{1}{c} = \frac{1}{v} + \mathrm{i}\frac{q}{\omega}, \tag{42}$$

then the wave number $k = R + \mathrm{i}q$, where $R = \omega/v$ and q are real numbers. This shows that v is the propagation speed and q is the attenuation coefficient of the surface waves (Fig. 1).

In Figs. 2–9, the graphical representation is given for two values of a dimensionless radius of the bore, $a = 1$ and $a = 5$ for comparison between an isotropic micropolar thermoelastic solid and a micropolar thermoelastic cubic crystal-like materials. The curves with dense horizontal lines and sparse horizontal lines represent variations for $a = 1$ and $a = 5$, respectively, for the isotropic case whereas the curves with a dense net and a sparse net represent variations for $a = 1$ and $a = 5$, respectively, for the anisotropic case. Figures 10–13 give the graphical representation for $a = 1$

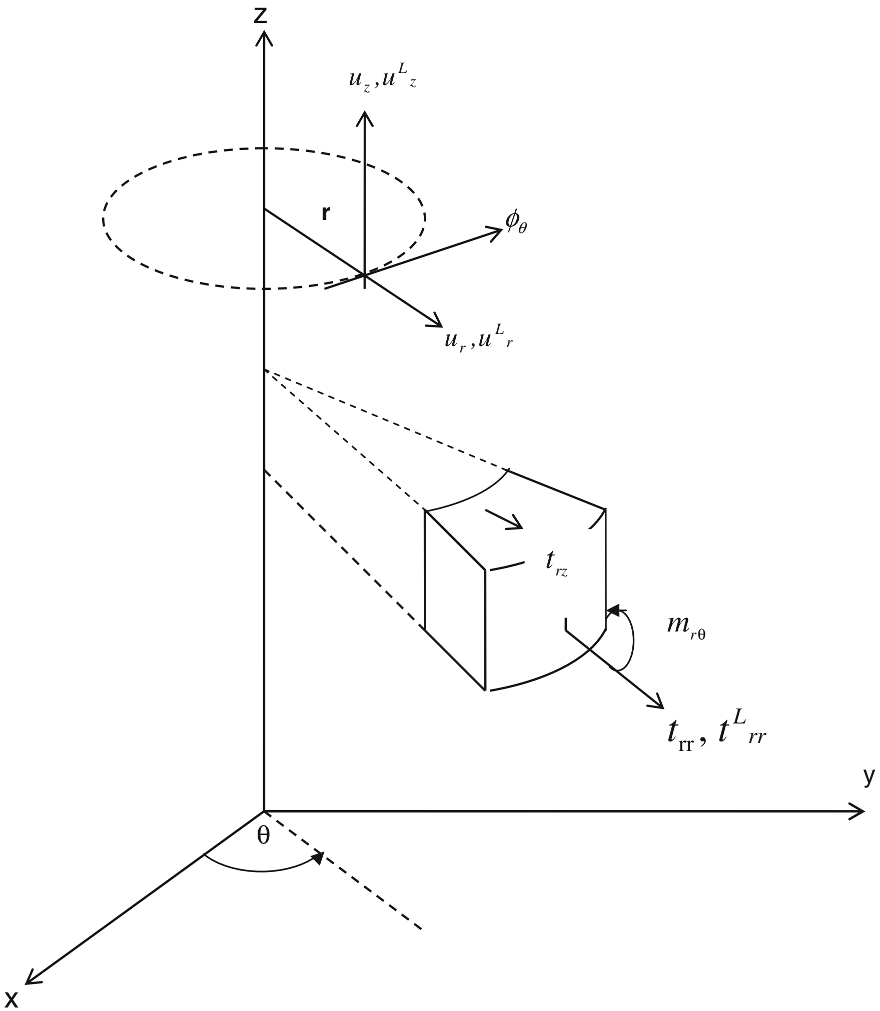


Fig. 1 Geometry of the investigated problem

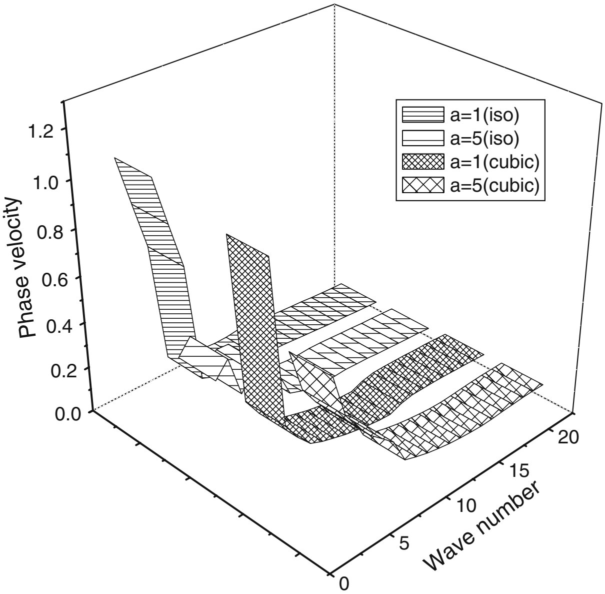


Fig. 2 Variation of phase velocity with respect to wave number for L–S theory in empty bore

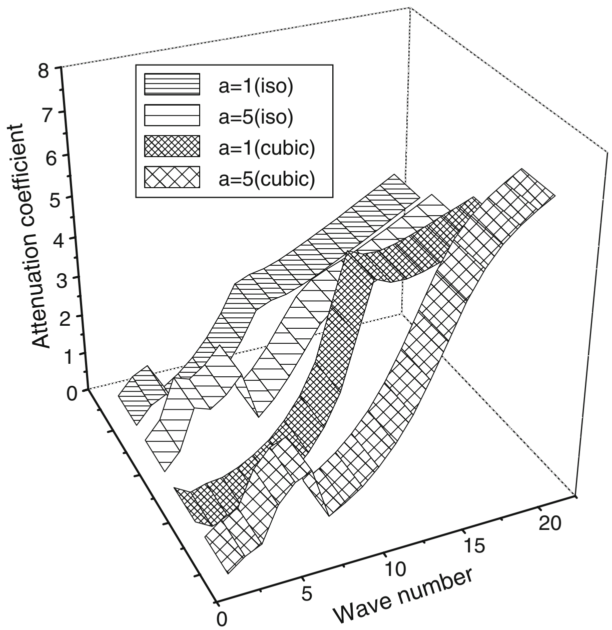


Fig. 3 Variation of attenuation coefficient with respect to wave number for L–S theory in empty bore

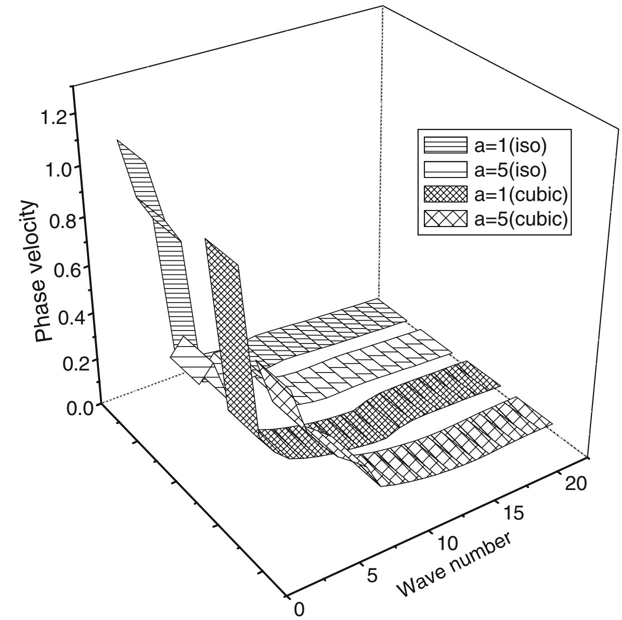


Fig. 4 Variation of phase velocity with respect to wave number for L–S theory in liquid-filled bore

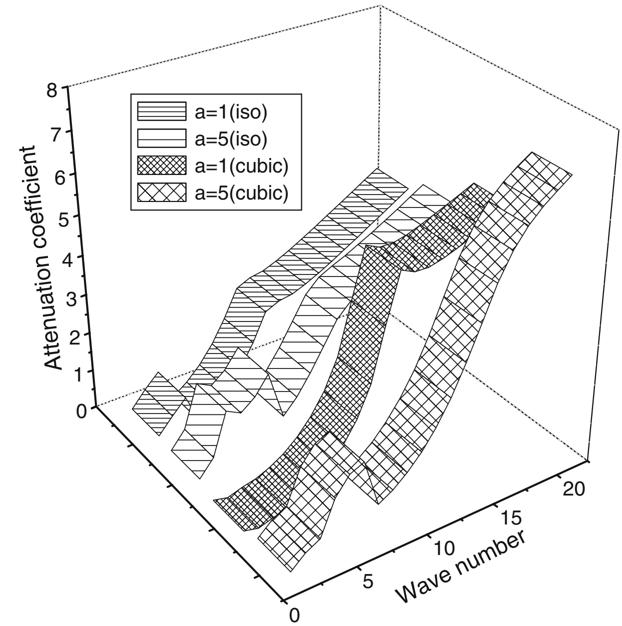


Fig. 5 Variation of attenuation coefficient with respect to wave number for L–S theory in liquid-filled bore

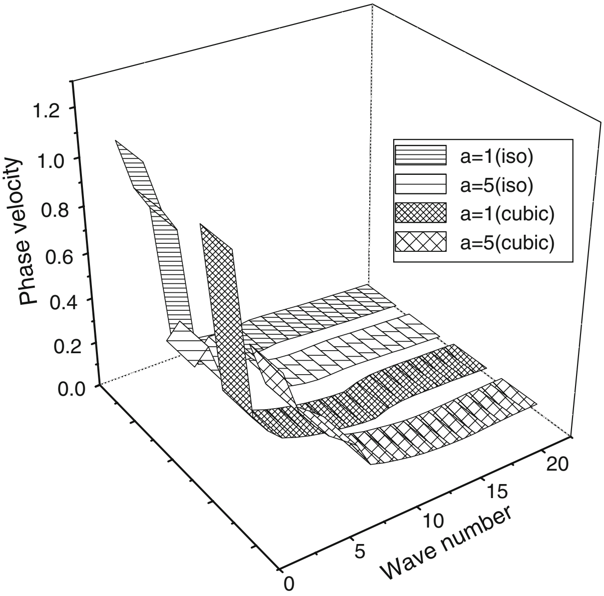


Fig. 6 Variation of phase velocity with respect to wave number for G-L theory in empty bore

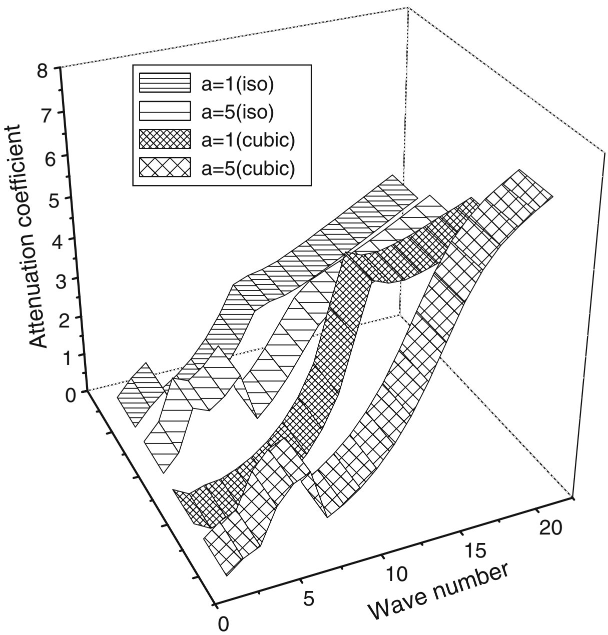


Fig. 7 Variation of attenuation coefficient with respect to wave number for G-L theory in empty bore

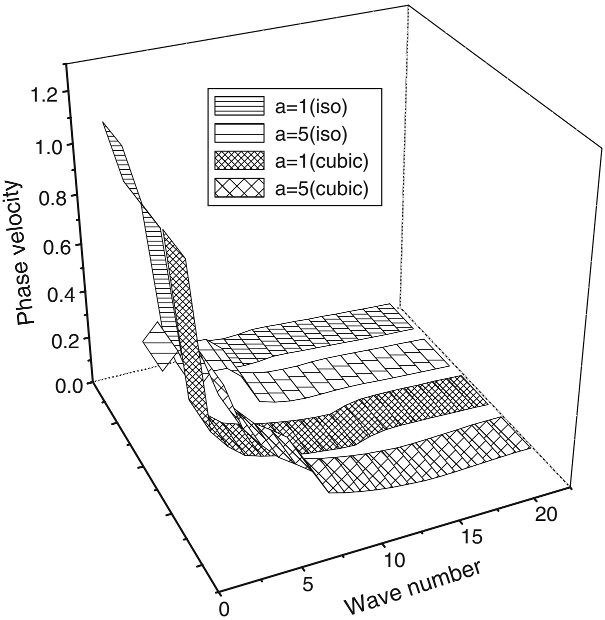


Fig. 8 Variation of phase velocity with respect to wave number for G–L theory in liquid-filled bore

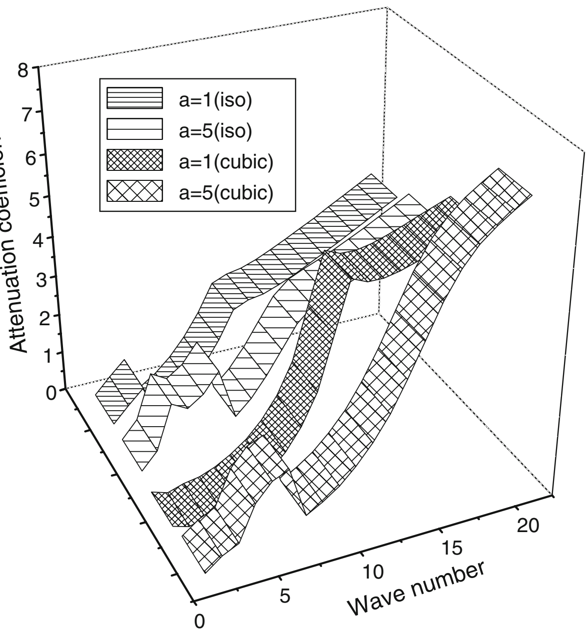


Fig. 9 Variation of attenuation coefficient with respect to wave number for G-L theory in liquid-filled bore

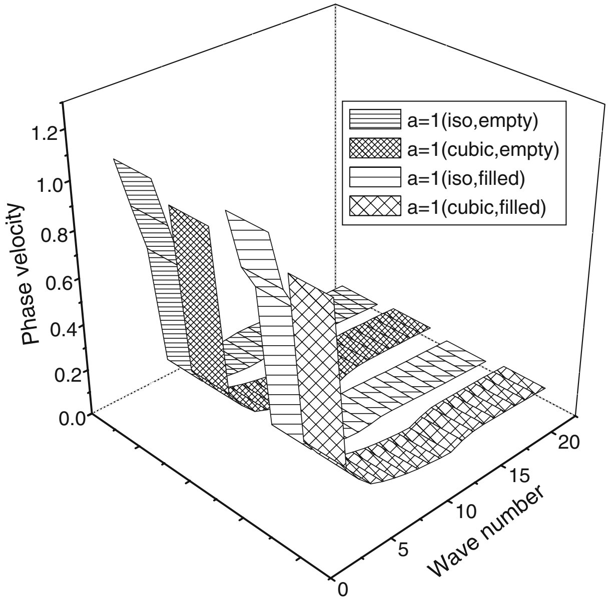


Fig. 10 Variation of phase velocity with respect to wave number for L–S theory in case of empty and liquid-filled bores for fixed radius

for comparison between variations for empty and liquid-filled bores. In these figures the curves with dense horizontal lines and sparse horizontal lines represent, respectively, the variations for empty and liquid-filled bores for the isotropic case whereas the curves with a dense net and a sparse net represent, respectively, the variations for empty and liquid-filled bores for the anisotropic case.

Figures 2 and 3 depict the variations of the phase velocity and attenuation coefficient with respect to R, i.e, the real part of the wave number for the L–S theory for the case of an empty bore whereas Figs. 4 and 5 represent the same situation for the case of a liquid-filled bore. Figures 6 and 7 depict the variations of the phase velocity and attenuation coefficient with respect to R, for the G–L theory for the case of an empty bore whereas Figs. 8 and 9 represent the same situation for the case of a liquid-filled bore. Figures 10 and 11 give the variation of the phase velocity and attenuation coefficient with respect to R, for empty and liquid-filled bores for the case of the L–S theory whereas Figs. 12 and 13 represent the same situation for the case of the G–L theory.

From Figs. 2, 4, 6, 8, and 12, it is observed that the phase velocity of wave propagation starts from a large value at a vanishing wave number and then exhibits strong dispersion at a higher wave number and ultimately attains a constant value. It is evident from Figs. 2, 4, and 6 that at initial values of the wave number and for $a = 1$, the anisotropic effect decreases the value of the phase velocity whereas the behavior is reversed for $a = 5$. Also, for the range $1 < R < 3$, the values of the phase velocity are higher for a smaller radius for the isotropic case whereas the behavior is reversed for the anisotropic case. Within the range $3 < R < 7$, the values of the phase velocity

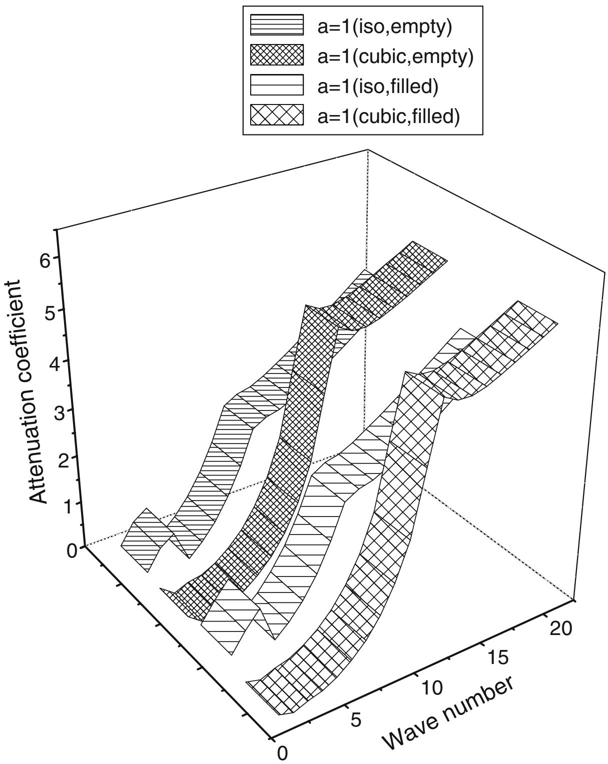


Fig. 11 Variation of attenuation coefficient with respect to wave number for L–S theory in case of empty and liquid-filled bores for fixed radius

Fig. 12 Variation of phase velocity with respect to wave number for G–L theory in case of empty and liquid-filled bores for fixed radius

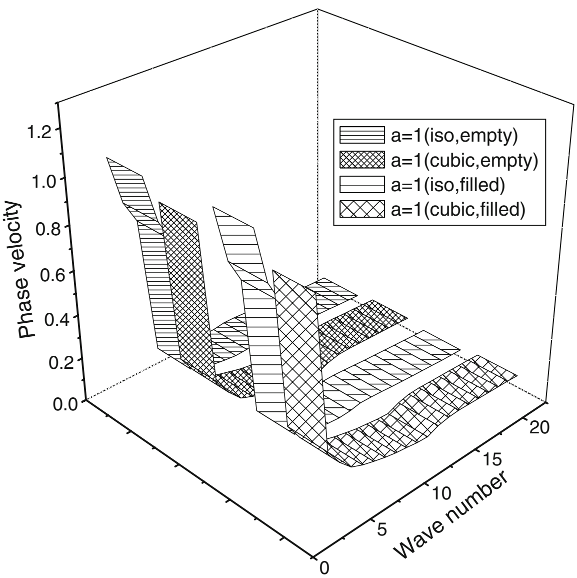
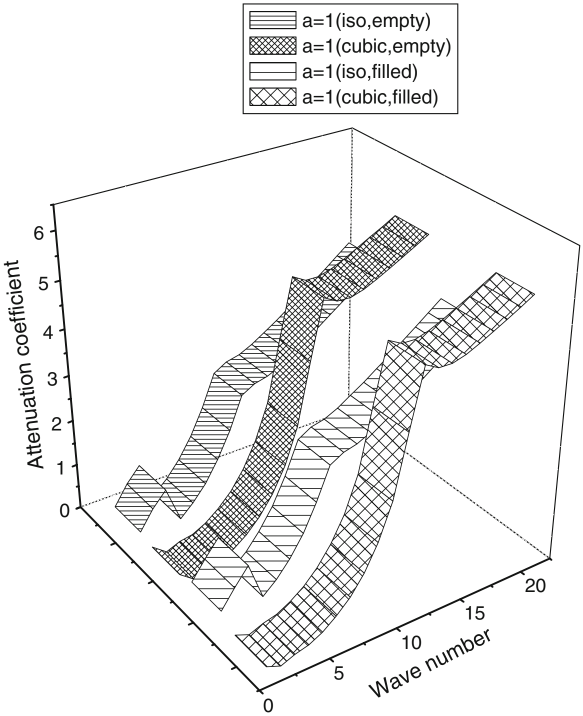


Fig. 13 Variation of attenuation coefficient with respect to wave number for G–L theory in case of empty and liquid-filled bores for fixed radius



are higher for $a = 5$ as compared to those for $a = 1$. At higher values of the wave number, all curves tend to attain constant values. In Figs. 10 and 12, the behavior and trend of variations of curves of the phase velocity is similar for empty and liquid-filled bores, but within the range $1 < R < 4$, values of the phase velocity are higher for the isotropic case as compared to those for the anisotropic case.

Unlike the phase velocity, the attenuation coefficient increases with R. It is observed from Figs. 3, 5, and 7 that within the ranges $1 < R < 7$ and $R > 10$, the values of the attenuation coefficient are higher for $a = 5$ than those for $a = 1$ for the isotropic case whereas the behavior is reversed for the anisotropic case. Within the range $7 < R < 10$, the curves show mixed behavior. Also, for $R > 15$, the values of the attenuation coefficient are higher for the anisotropic case as compared to those for the isotropic case. From Figs. 11 and 13, it is observed that the trend of variation of curves for empty and liquid-filled bores is almost similar, although magnitude values are slightly different.

However, within the range $1 < R < 10$, the values of the attenuation coefficient are higher for the isotropic case whereas the behavior is reversed for $R > 10$.

7 Conclusions

In this article, the Bessel function with complex arguments has been used to study wave propagation in a micropolar generalized thermoelastic medium possessing cubic symmetry. The phase velocity of the wave propagation has been computed from Eqs. 27 and 41 for different values of the wave number and for different boundary conditions. It is observed from all the graphs that the phase velocity decreases with the wave number and tends to attain a constant value at a higher wave number whereas the attenuation coefficient increases with the wave number. It is concluded that anisotropy has a significant impact on the phase velocity as well as on the attenuation coefficient.

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